

COUNTING CONICS IN COMPLETE INTERSECTIONS

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Abstract. We count the number of conics through two general points in complete intersections when this number is finite and give an application in terms of quasi-lines.

1. INTRODUCTION

Let X be a complex projective manifold of dimension n . A quasi-line l in X is a smooth rational curve $f : \mathbb{P}^1 \hookrightarrow X$ such that f^*T_X is the same as for a line in \mathbb{P}^n , i.e. is isomorphic to

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1}.$$

Let X be a smooth projective variety containing a quasi-line l . Following Ionescu and Voica [IV03], we denote by $e(X, l)$ the number of quasi-lines which are deformations of l and pass through two given general points of X . We denote by $e_0(X, l)$ the number of quasi-lines which are deformations of l and pass through a general point x of X with a given general tangent direction at x . Note that one always has $e_0(X, l) \leq e(X, l)$, but in general the inequality may be strict [IN03, p.1066].

1.1. Theorem. *Let $X \subset \mathbb{P}^{n+r}$ be a general smooth n -dimensional complete intersection of multi-degree (d_1, \dots, d_r) . Assume moreover that*

$$d_1 + \dots + d_r = \frac{n+1}{2} + r.$$

Then

- (1) *the family of conics contained in X is a nonempty, smooth and irreducible component of the Chow scheme $\mathcal{C}(X)$,*
- (2) *a general conic C contained in X is a quasi-line of X and*

$$e_0(X, C) = e(X, C) = \frac{1}{2} \prod_{i=1}^r (d_i - 1)! d_i!.$$

The numerical assumption $d_1 + \dots + d_r = (n+1)/2 + r$ assures that if C is a conic in X , then $-K_X \cdot C = n+1$. This numerical condition is of course necessary for a curve to be a quasi-line. Note that varieties appearing in our theorem are Fano varieties of dimension n and index $(n+1)/2$; they are well known to be the boundary Fano varieties with Picard number one being conic-connected (see [IR07], Theorem 2.2).

Using a degeneration argument, one can strengthen parts of the statement.

1.2. Corollary. *Let $X \subset \mathbb{P}^{n+r}$ be a smooth n -dimensional complete intersection of multi-degree (d_1, \dots, d_r) . If $d_1 + \dots + d_r = (n+1)/2 + r$, the variety X contains a conic that is a quasi-line.*

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By a theorem of Ionescu [Ion05], we obtain an immediate application of the theorem to formal geometry. Before stating it, let us recall that a subvariety Y of a variety X is G3 in X if the ring $K(X|_Y)$ of formal-rational functions of X along Y is equal to $K(X)$.

1.3. Corollary. *Let $X \subset \mathbb{P}^{n+r}$ be a general smooth n -dimensional complete intersection of multi-degree (d_1, \dots, d_r) such that $d_1 + \dots + d_r = (n+1)/2 + r$. Then any general conic C contained in X is G3 in X . In particular, if (X, C) and (X', C') are two such pairs such that the formal completions $X|_C$ and $X'|_{C'}$ are isomorphic as formal schemes, there exists an isomorphism from X to X' sending C to C' .*

When this note was almost finished, we learned from L. Manivel that A. Beauville had

obtained the formula $e(X, l) = \frac{1}{2} \prod_{i=1}^r (d_i - 1)! d_i!$ as a consequence of his computation

of the quantum cohomology algebra $H^*(X, \mathbb{Q})$ of a complete intersection [Bea95]. We provide here a completely elementary proof. We end this note by mentioning a similar question where no elementary proof seems to be known.

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2. PROOFS

We start by explaining the enumerative argument in the simplest case.

2.1. A well known example. Suppose that $X = \{s = 0\}$ is a smooth cubic threefold in \mathbb{P}^4 . A general conic C in X is a quasi-line [BBI00, Thm.3.2]. The basic idea of our proof is that counting conics in X through p and q can be reduced to counting 2-planes π through p and q such that the restriction $s|_{\pi}$ is a product of a polynomial of degree two and some residual polynomial. We will explain how to do this in general below, in the case of the cubic threefold we can use a geometric construction.

It is a classical fact that the lines in X form an irreducible smooth family of dimension two and that there are exactly six lines passing through a general point of X [AK77, Prop.1.7]. Fix now two general points p and q in X , then the line $[pq]$ intersects X in a third point u . For every line $l \subset X$ through u there exists a unique plane π_l containing l and $[pq]$. The intersection $X \cap \pi_l$ is the union of l and a residual conic C . Since l does not pass through p and q , the conic C passes through p and q . *Vice versa* the linear span of a conic $C \subset X$ passing through p and q is a 2-plane π_C containing the line $[pq]$. Since C does not pass through u , the residual line passes through u . Thus the conics through p and q are in bijection with the lines through u , so $e(X, C) = 6$.

Suppose now that we are in the general situation of Theorem 1.1. We always assume that $X \subset \mathbb{P}^{n+r}$ is a general smooth n -dimensional complete intersection of multi-degree (d_1, \dots, d_r) with $d_i \geq 2$ for all i and

$$d_1 + \dots + d_r = \frac{n+1}{2} + r.$$

Let $l \subset X$ be a smooth rational curve contained in X . Then

$$-K_X \cdot l = (n+r+1 - (d_1 + \dots + d_r)) \deg(l) = \frac{n+1}{2} \deg(l)$$

therefore $-K_X \cdot l = n + 1$ if and only if l is a conic.

2.2. The main step. *For any general points p and q of X , there exists a conic contained in X passing through p and q .*

Fix two distinct points in \mathbb{P}^{n+r} , say $p = [1 : 0 : \dots : 0]$ and $q = [0 : 0 : \dots : 1]$. Suppose that X is a general complete intersection with equations

$$(s_1 = 0) \cap (s_2 = 0) \cap \dots \cap (s_r = 0)$$

passing through p and q , where each $s_i \in H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d_i))$ is general among sections vanishing at p and q .

Suppose there is a conic C contained in X , passing through p and q and let π_C the projective 2-plane generated by C . If s_C denotes the equation defining C in π_C , there exists for each $i = 1, \dots, r$ a $\tilde{s}_i \in H^0(\pi_C, \mathcal{O}_{\pi_C}(d_i - 2))$ (defining the residual curve) such that

$$(s_i)|_{\pi_C} = s_C \cdot \tilde{s}_i.$$

Since X is general, it does not contain the 2-plane π_C [DM98, Thm. 2.1]. Therefore $(s_i)|_{\pi_C}$ and \tilde{s}_i are not zero for at least one i .

Conversely, let π be a projective 2-plane containing p and q and assume there exists a non-zero $s_C \in H^0(\pi, \mathcal{O}_{\pi}(2))$ vanishing at p and q and, for each $i = 1, \dots, r$, there exists a $\tilde{s}_i \in H^0(\pi, \mathcal{O}_{\pi}(d_i - 2))$ such that

$$(s_i)|_{\pi} = s_C \cdot \tilde{s}_i,$$

then the conic $(s_C = 0)$ is obviously contained in X .

Consider now the projective space of dimension $n + r - 2$ parametrizing the projective 2-planes in \mathbb{P}^{n+r} containing p and q . Fixing homogeneous coordinates $[a_1 : \dots : a_{n+r-1}]$ on this space, let

$$\pi_{[a_1 : \dots : a_{n+r-1}]} = \{[x : za_1 : \dots : za_{n+r-1} : y] \mid [x : z : y] \in \mathbb{P}^2\}$$

be such a 2-plane. Then

$$(s_i)|_{\pi_{[a_1 : \dots : a_{n+r-1}]}}(x, z, y) = \sum_{k=0}^{d_i} \sum_{a=0}^k s_{a,k}^i x^a y^{k-a} z^{d_i-k}$$

where $s_{a,k}^i$ is a homogeneous polynomial of degree $d_i - k$ in the variables a_1, \dots, a_{n+r-1} . The equation of an irreducible conic in this plane that passes through p and q is

$$s_C = s_2 z^2 + s_1 xz + s'_1 yz + xy.$$

So for each $i = 1, \dots, r$, the equation $(s_i)|_{\pi} = s_C \cdot \tilde{s}_i$ can be written explicitly

$$\sum_{k=0}^{d_i} \sum_{a=0}^k s_{a,k}^i x^a y^{k-a} z^{d_i-k} = (s_2 z^2 + s_1 xz + s'_1 yz + xy) \times \sum_{k=0}^{d_i-2} \sum_{a=0}^k \tilde{s}_{a,k}^i x^a y^{k-a} z^{d_i-2-k}.$$

Thus we have to solve the equations

$$s_{a,k}^i = s_2 \tilde{s}_{a,k}^i + s_1 \tilde{s}_{a-1,k-1}^i + s'_1 \tilde{s}_{a,k-1}^i + \tilde{s}_{a-1,k-2}^i$$

for any $0 \leq k \leq d_i$ and $0 \leq a \leq k$.

Let us first solve this system (whose unknown variables are s_2, s_1, s'_1 defining the conic and the $\tilde{s}_{a,k}^i$'s defining the residual curve) for each i separately. Note that X passes

through p and q if and only if $s_{0,d_i}^i = s_{d_i,d_i}^i = 0$. Therefore writing the $d_i - 1$ equations $s_{a,d_i}^i = \tilde{s}_{a-1,d_i-2}^i$ for $1 \leq a \leq d_i - 1$ provides the \tilde{s}_{a-1,d_i-2}^i 's.

Considering the equations corresponding to $(a, k) = (0, d_i - 1)$ and $(d_i - 1, d_i - 1)$ allows to find s_2 and s_1 . Considering then the equations corresponding to $(a, k) = (1, d_i - 1)$ and $(0, d_i - 2)$ gives \tilde{s}_{0,d_i-3}^i and s'_1 (in particular this determines the conic, if it exists!). Write down successively the equations for (a, k) , $a = 1, \dots, k - 1$, $k = d_i - 1, \dots, 2$ to find all the $\tilde{s}_{a,k}^i$'s (this determines the residual curve $(\tilde{s}^i = 0)$!).

Therefore, the r systems have a common solution if and only if the remaining equations for each system are satisfied and the corresponding conic is the same for each i . For each i , the remaining equations are “universal formulas” (meaning the coefficients just depend on the equations defining X) corresponding to $(a, k) = (0, d_i - 3), \dots, (0, 0)$ and $(a, k) = (d_i - 2, d_i - 2), \dots, (1, 1)$. This gives $2d_i - 4$ equations of respective degrees $3, \dots, d_i$ and $2, 3, \dots, d_i - 1$ in the variables a_1, \dots, a_{n+r-1} . The $3r - 3$ equations saying that the conic is the same for each $i = 1, \dots, r$ are $2r - 2$ equations of degree 1 and $r - 1$ equations of degree 2 in the variables a_1, \dots, a_{n+r-1} .

Altogether, using the relation $d_1 + \dots + d_r = (n + 1)/2 + r$, this gives exactly $n + r - 2$ homogeneous equations in the variables a_1, \dots, a_{n+r-1} . We therefore get at least one solution. Moreover since X is general, the coefficients $s_{a,k}^i$ appearing in the initial equations are general. Since they completely determine the remaining $n + r - 2$ homogeneous equations, these equations are general. Thus the space of solutions is

smooth and of the expected dimension, so there are exactly $\frac{1}{2} \prod_{i=1}^r (d_i - 1)! d_i!$ solutions by Bezout's theorem.

Let us briefly indicate how the same method gives *the number of conics contained in X , passing through p and tangent to the line (pq)* . With the above notations, we have $s_{d_i-1,d_i}^i = s_{d_i,d_i}^i = 0$ and we have to solve the r systems

$$\sum_{k=0}^{d_i} \sum_{a=0}^k s_{a,k}^i x^a y^{k-a} z^{d_i-k} = (s_2 z^2 + s_1 xz + s'_1 yz + y^2) \times \sum_{k=0}^{d_i-2} \sum_{a=0}^k \tilde{s}_{a,k}^i x^a y^{k-a} z^{d_i-2-k}$$

which means

$$s_{a,k}^i = s_2 \tilde{s}_{a,k}^i + s_1 \tilde{s}_{a-1,k-1}^i + s'_1 \tilde{s}_{a,k-1}^i + \tilde{s}_{a,k-2}^i$$

for any $0 \leq k \leq d_i$ and $0 \leq a \leq k$. The remaining details are left to the reader.

2.3. The space of conics is irreducible. Let $\mathbb{G}(2, n + r)$ be the Grassmannian of projective 2-planes contained in \mathbb{P}^{n+r} and E be the tautological rank 3-bundle on $\mathbb{G}(2, n + r)$. The Hilbert scheme of conics in \mathbb{P}^{n+r} is the projectivisation² of $S^2 E^*$. Denote by $\varphi : \mathbb{P}(S^2 E^*) \rightarrow \mathbb{G}(2, n + r)$ the natural map. We have an exact sequence on $\mathbb{P}(S^2 E^*)$:

$$(*) \quad 0 \rightarrow \bigoplus_{i=1}^r \varphi^* S^{d_i-2} E^* \otimes \mathcal{O}_{\mathbb{P}(S^2 E^*)}(-1) \rightarrow \bigoplus_{i=1}^r \varphi^* S^{d_i} E^* \rightarrow \mathcal{Q} \rightarrow 0$$

defining a vector bundle \mathcal{Q} of rank $n + 1 + 3r$. Since X is a complete intersection $(s_1 = 0) \cap (s_2 = 0) \cap \dots \cap (s_r = 0)$, the s_i 's induce by restriction to 2-planes, pull-back and projection onto \mathcal{Q} a section of \mathcal{Q} whose zero locus Z is precisely the set of conics

² In this article we follow the convention that the projectivisation of a vector bundle E is the variety of lines of E .

contained in X . Since E^* is globally generated, the images of sections (s_1, \dots, s_r) give a vector space $V \subseteq H^0(\mathbb{P}(S^2 E^*), \mathcal{Q})$ that globally generates \mathcal{Q} . Applying Bertini's theorem to this subspace we see that the zero locus of a general section in V is smooth. Since X is supposed to be a *general* complete intersection, Z is smooth and proving its irreducibility reduces to showing that $h^0(Z, \mathcal{O}_Z) = 1$. By the Koszul resolution of \mathcal{O}_Z , it is enough to show that for any $1 \leq j \leq \text{rk } \mathcal{Q}$

$$h^j(\mathbb{P}(S^2 E^*), \wedge^j \mathcal{Q}^*) = 0.$$

Using the exact sequence $(*)$, this easily reduces to showing that for any $1 \leq j \leq \text{rk } \mathcal{Q}$ and any $0 \leq k \leq j$,

$$H^k(\mathbb{P}(S^2 E^*), \wedge^k (\bigoplus_{i=1}^r \varphi^* S^{d_i} E) \otimes S^{j-k}((\bigoplus_{i=1}^r \varphi^* S^{d_i-2} E) \otimes \mathcal{O}_{\mathbb{P}(S^2 E^*)}(1))) = 0.$$

Since the higher direct images with respect to φ vanish, it is sufficient to show that for any $1 \leq j \leq \text{rk } \mathcal{Q}$ and for any $0 \leq k \leq j$, we have

$$H^k(\mathbb{G}(2, n+r), \wedge^k (\bigoplus_{i=1}^r S^{d_i} E) \otimes S^{j-k}((\bigoplus_{i=1}^r S^{d_i-2} E) \otimes S^2 E)) = 0.$$

This will follow from Bott's theorem applied on $\mathbb{G}(2, n+r)$. Indeed, using Schur functor notation, let $S_b E$ be an irreducible factor appearing in the decomposition of $\wedge^k (\bigoplus_{i=1}^r S^{d_i} E) \otimes S^{j-k}((\bigoplus_{i=1}^r S^{d_i-2} E) \otimes S^2 E)$ where $b = (b_1, b_2, b_3)$ is a triple of integers $b_1 \geq b_2 \geq b_3 \geq 0$. By the Littlewood-Richardson rule, we get

$$(**) \quad b_2 + b_3 \geq k - r \text{ and } b_3 \geq k - (d_1 + \dots + d_r) - r = k - (n+1)/2 - 2r.$$

On the other hand by Bott's theorem, the whole cohomology of $S_b E$ vanishes except maybe in the following cases :

- (1) $k = n+r-2$ and $(b_1, b_2, b_3) = (b_1, 0, 0)$ with $b_1 \geq n+r-1$,
- (2) $k = n+r-2$ and $(b_1, b_2, b_3) = (b_1, 1, 0)$ with $b_1 \geq n+r-1$,
- (3) $k = n+r-2$ and $(b_1, b_2, b_3) = (b_1, 1, 1)$ with $b_1 \geq n+r-1$,
- (4) $k = 2(n+r-2)$ with $b_2 \geq n+r$ and $b_3 = 0, 1, 2$,
- (5) $k = 3(n+r-2)$ with $b_3 \geq n+r+1$.

The case $n = 3$ has been dealt with by Bădescu, Beltrametti and Ionescu [BBI00], so we may assume $n \geq 5$ since n is odd.

In the first three cases, we get $k - r = n - 2 \geq 3 > b_2 + b_3 = 0, 1, 2$, which is excluded by $(**)$. In case (4), since $n \geq 5$, we get $k - (n+1)/2 - 2r = 3(n-3)/2 > b_3 = 0, 1, 2$, which is again excluded by $(**)$. Case (5) is also excluded since we are only interested in the situation where $k \leq \text{rk } \mathcal{Q} = n+1+3r$, but $3(n+r-2) > n+1+3r$ when $n \geq 5$.

We obtain the following corollary of the proof.

2.1. Corollary. *Let $X \subset \mathbb{P}^{n+r}$ be a general smooth n -dimensional complete intersection of multi-degree (d_1, \dots, d_r) . Assume moreover that*

$$d_1 + \dots + d_r \leq \frac{n+1}{2} + r$$

and $n \geq 5$. Then the family of conics contained in X is a nonempty, smooth and irreducible component of the Chow scheme $\mathcal{C}(X)$.

Let us also mention that Harris, Roth and Starr have shown the irreducibility of the space of smooth rational curves of arbitrary degree e for general hypersurfaces of low degree d [HRS04].

2.4. Conics are quasi-lines. By the first step, there exists a conic C passing through two general points. Such a conic is necessarily smooth: a line d contained in X and passing through a general point satisfies

$$T_X|_d \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \frac{n-3}{2}} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \frac{n+1}{2}},$$

so an easy dimension count shows that two general points are not connected by a chain of two lines. Thus C smooth and its deformations with a fixed point cover a dense open subset in X . This implies that the normal bundle $N_{C/X}$ is ample [Deb01, Prop.4.10] and since $-K_X \cdot C = n + 1$, the curve C is a quasi-line.

2.5. Proof of the Corollary 1.3. The irreducibility of the variety of conics gives

$$e_0(X, l) = e(X, l) = \frac{1}{2} \prod_{i=1}^r (d_i - 1)! d_i!.$$

The equality $e_0(X, l) = e(X, l)$ implies that general conics are G3 in X [Ion05, Cor. 4.6], in particular [Ion05, Cor. 4.7, Cor.1.9] apply.

3. A SIMILAR QUESTION

Using exactly the same method as developed in §2.3, one can prove the following result, left to the reader.

3.1. Proposition. *Let $X_d \subset \mathbb{P}^{n+1}$ be a general smooth n -dimensional hypersurface of degree d . Then, for $n \geq 7$ and $d \leq n + 1$, the family of conics contained in X_d is a nonempty, smooth and irreducible component of dimension $3n - 2d + 1$ of the Chow scheme $\mathcal{C}(X_d)$.*

In the case of $d = n + 1$, there is a finite number of conics passing through a general point of X_{n+1} . Let us denote by N_{n+1} this number. It seems that there are no known elementary method to compute this number. A general formula comes from the calculation of some Gromov-Witten invariants using mirror symmetry and an ordinary differential equation introduced by Givental. The following lines were written while reading [JNS04] and [Jin05].

3.2. Proposition. (Coates, Givental - Jinzenji, Nakamura, Suzuki) *Let $X_n \subset \mathbb{P}^n$ be a general smooth hypersurface of degree n in \mathbb{P}^n . Let N_n be the number of conics passing through a general point of X_n . Then*

$$N_n = \frac{(2n)!}{2^{n+1}} - \frac{(n!)^2}{2}.$$

Let us briefly explain where this result comes from. If a, b, c et d are four integers, let $\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle_d$ be the Gromov-Witten invariant counting the number (possibly infinite) of rational curves of degree d contained in X_n and meeting 3 general subspaces of \mathbb{P}^n , of respective codimension a, b and c . When a, b or c are equal to 1, each such rational curve has to be counted d times since the intersection of a degree d curve intersects a general hyperplane in d points. Since a general line meets X_n in n points, we get that $N_n = \langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_{n-1} \rangle_2 / 4n$. In [Jin05] are introduced some constants $\tilde{L}_m^{n+1, n, d}$, called “structure constants of the quantum cohomology ring of X_n ”. They satisfy the following formula:

$$\sum_{m=0}^{n-1} \tilde{L}_m^{n+1, n, 1} w^m = n \prod_{j=1}^{n-1} (jw + (n - j))$$

and

$$\sum_{m=0}^{n-2} \tilde{L}_m^{n+1,n,2} w^m = \sum_{j_2=0}^{n-2} \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} \tilde{L}_{j_1}^{n+1,n,1} \tilde{L}_{j_2+1}^{n+1,n,1} w^{j_1-j_0} \left(\frac{1+w}{2} \right)^{j_2-j_1}.$$

It is also shown in [Jin05] that for every integer m , $0 \leq m \leq n-2$, we have $\tilde{L}_m^{n+1,n,2} = \langle \mathcal{O}_1 \mathcal{O}_{n-1-m} \mathcal{O}_{m+1} \rangle_2 / n$.

Then the proposition follows by evaluating the w^{n-2} coefficient in the second formula above, the w^{n-1} coefficient in the first and putting $w = 2$.

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